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Continued Fractions and Fermionic Representations for Characters of $M(p,p')$ Minimal models

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Dedicated to Prof. Vladimir Rittenberg on his 60th birthday

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Abstract

We present fermionic sum representations of the characters $\chi_{r,s}^{(p,p')}$ of the minimal $M(p,p')$ models for all relatively prime integers $p'>p$ for some allowed values of r and s. Our starting point is binomial (q-binomial) identities derived from a truncation of the state counting equations of the XXZ spin $\frac{1}{2}$ chain of anisotropy $-\Delta=-\cos(\pi\frac{p}{p'})$. We use the Takahashi-Suzuki method to express the allowed values of r (and s) in terms of the continued fraction decomposition of $\{\frac{p'}{p}\}$ (and $\frac{p}{p'}$) where $\{x\}$ stands for the fractional part of x. These values are, in fact, the dimensions of the hermitian irreducible representations of $SU_{q_-}(2)$ (and $SU_{q_+}(2)$) with $q_-=\exp(i\pi\{\frac{p'}{p}\})$ (and $q_+=\exp(i\pi\frac{p}{p'})$). We also establish the duality relation $M(p,p')\leftrightarrow M(p'-p,p')$ and discuss the action of the Andrews-Bailey transformation in the space of minimal models. Many new identities of the Rogers-Ramanujan type are presented.

1. Introduction

Fermionic representations of conformal field theory characters are q series of the form

$$\sum_{\mathbf{m},\text{restrictions}} q^{\mathbf{m}\frac{\mathbf{B}}{2}\mathbf{m}+A\mathbf{m}} \prod_{a=1}^{n} \begin{bmatrix} ((1-\mathbf{B})\mathbf{m}+\frac{\mathbf{u}}{2})_a \\ m_a \end{bmatrix}_q \tag{1.1}$$

where $\mathbf{m}$ is an n component vector of nonnegative integers which may be subject to restrictions in the sum, $\mathbf{B}$ is an $n\times n$ matrix, $\mathbf{A}$ and $\mathbf{u}$ are n component vectors, and the q binomial coefficients (Gaussian polynomials) are defined as

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q)_n}{(q)_m(q)_{n-m}} \tag{1.2}$$

where

$$(q)_n=\prod_{k=1}^{n}(1-q^k) \tag{1.3}$$

and we note

$$\lim_{n\to\infty} \begin{bmatrix} n \\ m \end{bmatrix} = \frac{1}{(q)_m}. \tag{1.4}$$

These representations are built from a representation of the space of states in terms of n quasi particles which obey a Pauli momentum exclusion rule with momentum ranges that depend linearly on the number of particles $\mathbf{m}$ in the state [1]-[6]. They are in contrast to bosonic representations of the characters [7]-[13] based upon a truncation of a bosonic Fock

space where there is no direct quasi-particle interpretation for the states. The bosonic representations are best adapted for ultraviolet, short distance properties while the fermionic representations are best adapted for infrared, long distance properties. The bosonic representations are in general unique whereas there are usually several different fermionic representations which correspond to the different massive integrable perturbations which can be put on the conformal field theory. The equality of the fermionic and bosonic representations is a generalization of the one hundred year old identities of Rogers-Schur-and Ramanujan [14]-[17].

The theory of bosonic representations is very well developed. However, the corresponding fermionic representations are still under investigation. The first such representation discovered was for the Z_N parafermionic models [18] and in the last two years a large number of fermionic representations of characters and branching functions for affine Lie algebras of integer level have been found [1]-[6], [25]- [34].

However, what is probably the best known example of conformal field theory, the $M(p,p')$ minimal models of Belavin, Polyakov, and Zamolodchikov [35] is not in general of this type if $p \neq p'+1$. There are at least two ways to see this:

1. The coset construction [36] of fractional level [37]-[39].

Here we represent $M(p,p')$ as

$$\frac{(A_1^{(1)})_1 \otimes (A_1^{(1)})_m}{(A_1^{(1)})_{m+1}} \tag{1.5}$$

where the level m is

$$m = \frac{p}{p'-p} - 2 \quad \text{or} \quad -\frac{p'}{p'-p} - 2. \tag{1.6}$$

2. The Hamiltonian reduction method [40]-[41].

The origin of this method is in the paper of Drinfeld and Sokolov [42]. In this context the characters of $M(p,p')$ are obtained [43] as the "residue" of the character of $(A_1^{(1)})_m$ where the level is

$$m = \frac{p}{p'} - 2 \quad \text{or} \quad \frac{p'}{p} - 2. \tag{1.7}$$

All four of these fractional levels will be seen in the fermionic representations presented below.

The bosonic form of this character is the well known result of Rocha-Caridi [8]

$$\hat{\chi}_{r,s}^{(p,p')} = q^{\Delta_{r,s}^{(p,p')} - c/24} \chi_{r,s}^{(p,p')} \tag{1.8}$$

where

$$\chi_{r,s}^{(p,p')} = \frac{1}{(q)_\infty} \sum_{k=-\infty}^{\infty} (q^{k(kpp'+rp'-sp)} - q^{(kp'+s)(kp+r)}), \tag{1.9}$$

the conformal dimensions are

$$\Delta_{r,s}^{(p,p')} = \frac{(rp'-sp)^2-(p-p')^2}{4pp'} \qquad (1 \leq r \leq p-1,\ 1 \leq s \leq p'-1), \tag{1.10}$$

the central charge is

$$c = 1 - \frac{6(p-p')^2}{pp'}. \tag{1.11}$$

For the minimal unitary model $p' = p+1$ the fermionic form of $\chi_{r,s}^{(p,p')}$ has given by [3] and proven in [29]. When $p' \neq p+1$ fermionic forms have been found for the following cases:

1. $\chi_{1,s}^{(2,2n+1)}$ for $n = 2, 3, \cdots$ and all s [44]. The equality of the bose and fermi forms are the original identities of Rogers-Schur-Ramanujan [14]-[17] when $n=2$ and are the Andrews-Gordon identities [45]-[46] for $n \geq 3$.
2. $\chi_{1,kj}^{(p,kp+1)}$ and $\chi_{1,k(j+1)+1}^{(p,kp+1)}$ for $p \geq 4, 1 \leq j \leq p-2$ and $k \geq 1$ [3] [31];
3. $\chi_{1,j(k+1)}^{(p,kp+p-1)}$ and $\chi_{1,j(k+1)+k}^{p,kp+p-1)}$ for $p \geq 4, 1 \leq j \leq p-2$ and $k \geq 1$ [31];
4. $\chi_{s,kn+1}^{(2n+1,k(2n+1)+2)}$ and $\chi_{s,(k+1)n+1}^{(2n+1,k(2n+1)+2)}$ for $1 \leq s \leq k$ [3][31];
5. $\chi_{s,kn+k-1}^{(2n+1,k(2n+1)+2n-1)}$ and $\chi_{s,(k+1)n+k-1}^{(2n+1,k(2n+1)+2n-1)}$ for $1 \leq s \leq k$ [31].

In this paper we generalize these results for characters of $M(p,p')$ minimal models to all p and p'. Our method will be a generalization of [29]. We will first formulate in sec. 2 a fermionic state counting problem for the $M(p,p')$ system with a finite number of states and relate it to a bosonic counting formula. We will then generalize these fermi-bose counting identities to q deformed identities for the finite system which in the infinite limit give identities of fermionic forms (1.1) with the Rocha-Caridi bosonic character (1.9). We consider separately the cases $p' > 2p$ in sec. 3 and $p' < 2p$ in sec. 4 and conclude with discussions in sec. 5.

2. Counting and continued fractions for $p' > 2p$.

The counting formulae for the general minimal models are obtained by an extension of the counting techniques of [29] which were developed for the unitary models $p' = p+1$.

This counting problem for $M(p,p')$ is obtained from the thermodynamic treatment of the XXZ spin chain

$$H_{XXZ} = -\sum_k \left(\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \Delta \sigma_k^z \sigma_{k+1}^z\right) \tag{2.1}$$

where σ_k^i $(i = x, y, z)$ are Pauli spin matrices and

$$\Delta = -\cos \pi \frac{p}{p'} \tag{2.2}$$

which was solved by Takahashi and Suzuki in 1972 [47].

To present the counting solution when $p' > 2p$ we define the $n+1$ positive integers ν_j from the continued fraction decomposition of p'/p

$$\frac{p'}{p} = (\nu_0 + 1) + \frac{1}{\nu_1 + \frac{1}{\nu_2 + \cdots + \frac{1}{\nu_n + 2}}}. \tag{2.3}$$

For $p' > 2p$ we have $\nu_0 \geq 1$ and we will tacitly assume $\nu_n \geq 0$. The case $\nu_n = -1$ follows with only a slight change in notation. The interval

$$0 \leq k \leq \sum_{j=0}^{n} \nu_j \tag{2.4}$$

is divided into $n+1$ subintervals $(i = 1, \cdots, n+1)$

$$1 + t_{i-1} \leq k \leq t_i \tag{2.5}$$

where

$$t_i = \sum_{j=0}^{i-1} \nu_j, \quad \text{for} \quad i = 1, \cdots, n+1 \quad \text{and} \quad t_0 = -1. \tag{2.6}$$

We refer to the index j of ν_j as the "zone" index, ν_j as the number of bands in the j^{th} zone, t_{i+1} is the upper end of the zone i, and we say that when (2.5) holds then k lies in the zone i (note that our t_i is $m_i - 1$ of [47]) . We also adopt the convention that when $\nu_n = 0$ we shall say that there is a zone n even though that zone contains no bands. The counting problem for the models $M(p,p')$ differs from the counting problem of the XXZ chain only in that in the XXZ chain the number of bands in the last n^{th} zone is $\nu_n + 2$ instead of ν_n. This reduction of the number of quasi-particles in the last zone is sometimes referred to as a truncation of the space of states and has been often used [48]-[49]in connection with the RSOS models [50].

From the ν_j we define the sets of integers y_i recursively as

$$y_{-1} = 0, \quad y_0 = 1, \quad y_1 = \nu_0 + 1, \quad y_{i+1} = y_{i-1} + (\nu_i + 2\delta_{i,n})y_i. \tag{2.7}$$

We then define what we refer to as the set of allowed string lengths l_k in each zone i as

$$l_k = y_{i-1} + [k - 1 - t_i]y_i \ \text{ for } \ 1 + t_i \leq k \leq t_{i+1} \tag{2.8}$$

and we note as examples

$$\begin{aligned} l_k &= k \text{ for } 0 \leq k \leq \nu_0 = t_1 \\ &= 1 + (\nu_0 + 1)(k - \nu_0 - 1) \ \text{ for } \ \nu_0 + 1 \leq k \leq \nu_0 + \nu_1 = t_2, \\ &= (\nu_0 + 1) + [(\nu_0 + 1)\nu_1 + 1](k - \nu_0 - \nu_1 - 1) \ \text{ for } \ t_2 + 1 \leq k \leq t_3. \end{aligned} \tag{2.9}$$

We also define a second set of integers z_i

$$z_{-1} = 0, \quad z_0 = 1, \quad z_1 = \nu_1, \quad z_{i+1} = z_{i-1} + (\nu_{i+1} + 2\delta_{i+1,n})z_i, \tag{2.10}$$

and a second set of string lengths $\tilde{l}_k$ by

$$\tilde{l}_k = z_{i-1} + [k - 1 - \tilde{t}_i]z_i \ \text{ for } \ 1 + \tilde{t}_i \leq k \leq \tilde{t}_{i+1} \tag{2.11}$$

where $\tilde{t}_i = t_{i+1} - \nu_0$. It is clear that the z_i are obtained from the same set of recursion relations as the y_i except that zone zero is removed in the partial fraction decomposition of p'/p. The removal of this zone zero is equivalent to considering a new XXZ chain with a anisotropy $\Delta' = -\cos\pi\{\frac{p'}{p}\}$ where $\{x\}$ denotes the fractional part of x.

It may now be proven by induction that

$$\frac{y_{i+1}}{z_i} = (\nu_0 + 1) + \cfrac{1}{\nu_1 + \cdots \frac{1}{\nu_i}.} \tag{2.12}$$

It is then clear from (2.3) that

$$p' = y_{n+1} \ \text{ and } \ p = z_n. \tag{2.13}$$

We may also prove by induction that

$$y_{i+1}z_{i-1} - y_i z_i = (-1)^{i-1}. \tag{2.14}$$

Then taking i to be n we obtain

$$(-1)^{n-1} = y_{n+1}z_{n-1} - y_n z_n = p' z_{n-1} - p y_n. \tag{2.15}$$

From the definition (1.10) we see that the minimal conformal dimension is obtained for the values $r_{\min}$ and $s_{\min}$ which satisfy

$$|p' r_{\min} - p s_{\min}| = 1. \tag{2.16}$$

and thus from (2.15) that

$$s_{\min} = y_n \text{ and } r_{\min} = z_{n-1}. \tag{2.17}$$

The counting problem is obtained from equations of the thermodynamic Bethe's Ansatz of the XXZ chain (eqn (3.9) of [47]) by taking the 0^{th} Fourier component and using the notation that the nonnegative integers n_j (m_j) are number of particle (hole) excitations of type j. Thus, with the truncation of the number of particles in the last (n^{th}) zone from $\nu_n + 2$ to ν_n we consider the following generalization of the equation (1.10) of [29]

$$\begin{aligned} n_k + m_k &= \frac{1}{2}(m_{k-1} + m_{k+1}) \text{ for } 1 \le k \le t_{n+1} - 1 \text{ and } k \neq t_i,\ i = 1, \cdots, n \\ n_{k_i} + m_{k_i} &= \frac{1}{2}(m_{k_i-1} + m_{k_i} - m_{k_i+1}) \text{ for } k_i = t_i,\ i = 1, \cdots, n \\ n_{t_{n+1}} + m_{t_{n+1}} &= \frac{1}{2}(m_{t_{n+1}-1} + m_{t_{n+1}}\delta_{\nu_n,0}) \end{aligned} \tag{2.18}$$

where by definition $m_0 = L$. (To simplify our presentation we assume throughout the rest of this section that $\nu_n \neq 0$.)

This construction of $M(p, p')$ out of the XXZ model is closely related to the representation of $M(p, p')$ in terms of two quantum groups [51] (and references therein)$SU_{q_\pm}(2)$ where

$$q_+ = e^{\frac{\pi i p}{p'}} \text{ and } q_- = e^{\frac{\pi i p'}{p}} \tag{2.19}$$

which is clearly related to the fractional levels of (1.7). In this quantum group construction the string lengths l_k and $\tilde{l}_k$ can be interpreted as the dimensions of the hermitian representations of $SU_{q_+}(2)$ ($SU_{q_-}(2)$) [52]-[55].

To proceed further we convert (2.18)into the following partition problem for L

$$\sum_{i=1}^{t_{n+1}} n_i l_i + \frac{m_{t_n+1}}{2} l_{(t_{n+1}+1)} = \frac{L}{2} \tag{2.20}$$

where $l_{(t_{n+1}+1)} = p' - 2y_n$. Depending on whether m_{t_n+1} is even or odd we consider two cases which we will refer to as "e" and "o" in what follows

$$\begin{aligned} &e \ \text{ is the case } \ m_{t_{n+1}} = 2n_{(t_{n+1}+1)} \\ &o \ \text{ is the case } \ m_{t_{n+1}} = 2n_{(t_{n+1}+1)} + 1 \end{aligned} \tag{2.21}$$

Given the set $(n_1, n_2, \cdots, n_{(t_{n+1}+1)})$ satisfying (2.20) we can solve (2.18) to obtain the companion set $\{m_1, m_2, \cdots, m_{t_{n+1}-1}\}$

The particles and holes in (2.18) are fermionic and thus for each n_j and m_j the number of distinct states in the band j is given by the binomial coefficient

$$\binom{n_j + m_j}{m_j} = \frac{(m_j + n_j)!}{m_j! n_j!}. \tag{2.22}$$

Thus the fermionic counting problem for the $M(p, p')$ models is the evaluation of

$$F(L)_{e,o} = \sum_{m_{t_{n+1}} = \text{even,odd}} \prod_{j=1}^{t_{n+1}} \binom{n_j + m_j}{n_j} \tag{2.23}$$

where the sum is with respect to all solutions of (2.20). Our result is that for L even

$$F(L)_e = B_{s_{\min}, s_{\min}}(L) \tag{2.24}$$

and for $L + p'$ even

$$F(L)_o = B_{s_{\min}, p' - s_{\min}}(L) \tag{2.25}$$

where the bose counting function that of Forrester and Baxter [56]

$$B_{a,b}(L) = \sum_{j=-\infty}^{\infty} \left[\binom{L}{\frac{L+a-b}{2} - jp'} - \binom{L}{\frac{L-a-b}{2} - jp'} \right]. \tag{2.26}$$

These counting formulae will be sufficient to give the characters $\chi^{(p,p')}_{r_{\min}, s_{\min}}$ and $\chi^{(p,p')}_{r_{\min}, p' - s_{\min}}$.

To obtain further characters we consider solutions to (2.18) with inhomogeneous terms. Thus if we denote (2.18) symbolically as

$$\mathbf{n} = \mathbf{M} \cdot \mathbf{m} + \frac{L}{2} \mathbf{e}_1 \tag{2.27}$$

where $(\mathbf{e}_k)_j = \delta_{j,k}$ we will consider the equations with an inhomogeneous term $\mathbf{u}$

$$\mathbf{n} = \mathbf{M} \cdot \mathbf{m} + \frac{L}{2}\mathbf{e}_1 + \frac{\mathbf{u}}{2} \tag{2.28}$$

and define from (2.23) the corresponding Fermi sum $F(l, \mathbf{u})_{e,o}$. In this paper we confine our attention to those inhomogeneous terms where the fermionic counting sum is equal to precisely one bosonic sum $B_{r,s}(L)$. These inhomogeneous terms are given in terms of what we will call a string configuration

$$\begin{aligned} \mathbf{u}_k^{(j)} &= \delta_{k,j} - \sum_{l=i}^{n} \delta_{k,t_l} \quad \text{for} \quad t_{i-1} < j \leq t_i \text{ and } i \leq n \\ &= \delta_{k,j} \text{ for } i = n+1 \end{aligned} \tag{2.29}$$

We will say that j is the endpoint of the string. An example is shown in table 1. We find that only one and two string configurations give a single bosonic sum. In particular

1. One string configurations

$$\begin{aligned} F(L, \mathbf{u}^{(j)})_e &= B_{a,s_{\min}}(L) \quad \text{with} \quad L + a + s_{\min} \text{ even} \\ F(L, \mathbf{u}^{(j)})_o &= B_{p'-a,s_{\min}}(L) \quad \text{with} \quad L + p' + a + s_{\min} \text{ even} \end{aligned} \tag{2.30}$$

with

$$a = l_{j+1} \quad \text{if} \quad t_i < j < t_{i+1} \quad \text{and} \quad y_{i+1} - 2y_n\delta_{i,n} \quad \text{if } j = t_{i+1} \tag{2.31}$$

where l_k is the string length (2.8).

2. Two string configurations.

The inhomogeneous term is now given as

$$\mathbf{u}^{(j_1,j_2)} = \mathbf{u}^{(j_1)} + \mathbf{u}^{(j_2)} \tag{2.32}$$

and we find

$$F(L, \mathbf{u}^{(j_1,j_2)})_o = B_{p'-a,b}(L) \quad \text{with } L + p' + a + b \text{ even} \tag{2.33}$$

where a (and b) are defined by (2.31) with j replaced by j_1 (and j_2).

We prove these results by a generalization of the fermionic counting methods of [29]. We will only sketch the prove here. Details will be presented elsewhere.

Define a generating function for the fermionic sum (2.23) with the n_j and m_j related by (2.28) with

$$\mathbf{u} = \sum_{j=1}^{t_{n+1}} c_j \mathbf{e}_j \tag{2.34}$$

as

$$G(x)_i = \sum_L x^{L/2} F(L,\mathbf{u})_i \tag{2.35}$$

where $i = e, o$ and c_j are some integers.

Then following [29] we can evaluate this generating function as

$$G(\theta)_i = G^{(0)}(\theta)_i \prod_{j=1}^{t_{n+1}} \left(\frac{U_j(\theta)}{U_1^{l_j}(\theta)} \right)^{c_j} \tag{2.36}$$

where $2\cos\theta = x^{-\frac{1}{2}}$ and

$$\begin{aligned} \frac{\sin\theta \sin(p'\theta)}{\sin(y_n\theta)} G^{(0)}(\theta)_i &= \sin(p'-y_n)\theta \quad \text{if} \quad i=e \\ &= \sin y_n\theta \quad \text{if} \quad i=o \end{aligned} \tag{2.37}$$

and the generalized Chebyshev polynomials $U_j(\theta)$ are

$$\begin{aligned} U_j(\theta) &= \frac{\sin(l_{j+1}\theta)}{\sin(y_i\theta)} \quad \text{for} \quad t_i < j < t_{i+1} \\ &= \frac{\sin((y_{i+1}-2y_n\delta_{i,n})\theta)}{\sin(y_i\theta)} \quad \text{for} \quad j = t_{i+1} \end{aligned} \tag{2.38}$$

The resulting contour integral expression for $F(L,\mathbf{u}))_i$ can be evaluated by means of residue calculus. When $\mathbf{u}$ is such that there is one pole which contributes to the integral the above results are obtained.

3. Characters for $p' > 2p$.

The principle of [29] is that the fermi-bose identities of characters are obtained by taking the $L \to \infty$ limit of polynomial identities which are obtained from the counting identities by the following "q-deformations"

1.

$$B_{a,b}(L) \to B_{a,b;r,s}(L,q) = \sum_{j=-\infty}^{\infty} \left[q^{j(jpp'+rp'-sp)} \begin{bmatrix} L \\ \frac{L+a-b}{2} - jp' \end{bmatrix}_q - q^{(jp+r)(jp'+s)} \begin{bmatrix} L \\ \frac{L-a-b}{2} - jp' \end{bmatrix}_q \right] \tag{3.1}$$

In the limit $L \to \infty$ we have $B_{a,b;r,s}(L,q) \to \chi_{r,s}^{(p,p')}(q)$. The function $B_{a,b;,r,s}(L,q)$ is that of [56].

The deformation of the fermionic sums $F(L,\mathbf{u})$ depends on which of n_j and m_j (related by (2.28)) are chosen as independent variables. For the present case we use as independent variables

$$\tilde{\mathbf{m}}^t = (n_1, \cdots, n_{\nu_0}, m_{\nu_0+1}, \cdots, m_{t_{n+1}}). \tag{3.2}$$

Then using (2.28) we define the following q deformation of $F(L,\mathbf{u})$

2.

$$F(L,\mathbf{u},q)_{e,o} =$$
$$q^{\Delta} \sum_{e,o,\text{restrictions}} q^{\frac{1}{2}\tilde{\mathbf{m}}\mathbf{B}\tilde{\mathbf{m}}+\mathbf{A}\tilde{\mathbf{m}}} \prod_{j=1}^{t_{n+1}} \begin{bmatrix} ((1-\mathbf{B})\tilde{\mathbf{m}} + L\sum_{l=1}^{\nu_0}\mathbf{e}_l + \frac{L}{2}\mathbf{e}_{\nu_0+1} + f(\mathbf{u}))_j \\ \tilde{m}_j \end{bmatrix}_q \tag{3.3}$$

where the e,o restrictions are defined below, Δ is chosen so that $F(L,\mathbf{u},0)=1$ and the nonzero elements of the matrix $\mathbf{B}$ are

$$\begin{aligned}
&B_{i,j} = 2\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & \dots & 2 \\ 1 & 2 & 3 & \dots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \dots & \nu_o \end{pmatrix} \quad \text{for } \ 1 \le i,j, \le \nu_0 \\
&B_{\nu_0+1,j} = B_{j,\nu_0+1} = j \ \text{ for } \ 1 \le j \le \nu_0 \\
&B_{j,j} = \frac{\nu_0}{2}\delta_{j,\nu_0+1} + (1-\frac{1}{2}\delta_{j,t_i}) \ \text{ for } \ \nu_0+1 \le j \le t_{n+1}-1 \text{ and } 2 \le i \le n \\
&B_{t_{n+1},t_{n+1}} = 1 - \frac{1}{2}\delta_{\nu_n,0} \\
&B_{j,j+1} = B_{j,j-1} = -\frac{1}{2} \ \text{ for } j \ne t_i \text{ and } 2 \le i \le n \\
&B_{j,j+1} = -B_{j,j-1} = \frac{1}{2} \ \text{for } j = t_i
\end{aligned} \tag{3.4}$$

and

$$f(\mathbf{u}) = \frac{1}{2}\mathbf{B}\mathbf{u}_+ + \frac{1}{2}\mathbf{u}_- \tag{3.5}$$

where we decompose $\mathbf{u} = \mathbf{u}_+ + \mathbf{u}_-$ with

$$\begin{aligned}
(\mathbf{u}_+)_i &= (\mathbf{u})_i \text{ for } i \le \nu_0 \text{ zero otherwise} \\
(\mathbf{u}_-)_i &= (\mathbf{u})_i \text{ for } i > \nu_0 \text{ zero otherwise.}
\end{aligned} \tag{3.6}$$

The linear terms and the even (odd) restrictions on $\tilde{m}_k$ are determined by the vector $\mathbf{u}$. The restrictions on m_j are uniquely determined from the vector $\mathbf{u}$ and the parity of

$m_{t_{n+1}}$ by the requirement that the numerator in the q-binomial coefficients in (3.3) be an integer. To describe the restrictions it is convenient to introduce the vector $v^{(j)}$ with $t_{l_0} < j \le t_{l_0+1} + \delta_{l_0,n}$ defined as follows

$$\begin{aligned} v_k^{(j)} &= (j-k) \text{ for } j > k \text{ and } k \ge t_{l_0} \\ v_k^{(j)} &= v_{k+1}^{(j)} + v_{t_{l+1}+1}^{(j)} \quad t_l \le k < t_{l+1}; \quad l < l_0. \end{aligned} \tag{3.7}$$

In terms of $v^{(j)}$ the restrictions can be written as

$$\begin{aligned} \text{even restriction} : m_i &= \sum_{j=1}^{t_{n+1}} v_i^{(j)} a_j \pmod 2 \\ \text{odd restriction} : m_i &= \sum_{j=1}^{t_{n+1}} v_i^{(j)} a_j + v_i^{(t_{n+1}+1)} \pmod 2 \end{aligned} \tag{3.8}$$

with $a_j = 1$ if $\mathbf{u}_j \neq 0$ and 0 otherwise.

Let us now proceed to define the linear terms. The simplest case is $\mathbf{u} = 0$. In this case $\mathbf{A} = 0$ and we have

$$\begin{aligned} F(L,0,q)_e &= B_{s_{\min},s_{\min};r_{\min},s_{\min}}(L,q) \\ F(L,0,q)_o &= B_{p'-s_{\min},s_{\min};p-r_{\min},s_{\min}}(L,q) \end{aligned} \tag{3.9}$$

Taking $L \to \infty$ in (3.9) we obtain

$$\chi_{r_{\min},s_{\min}}^{(p,p')} = q^{\Delta} \sum_{\tilde{m}_i = \text{even } (i > \nu_0)} q^{\frac{1}{2}\tilde{\mathbf{m}}\mathbf{B}\tilde{\mathbf{m}}} \prod_{j=1}^{\nu_0+1} \frac{1}{(q)_{\tilde{m}_j}} \prod_{j=\nu_0+2}^{t_{n+1}} \begin{bmatrix} ((1-\mathbf{B})\tilde{\mathbf{m}})_j \\ \tilde{m}_j \end{bmatrix}_q \tag{3.10}$$

and

$$\chi_{r_{\min},p'-s_{\min}}^{(p,p')} = q^{\Delta} \sum_{\tilde{m}_i = v_i^{(t_{n+1}+1)} (\mathrm{mod} 2)(\text{for } i > \nu_0)} q^{\frac{1}{2}\tilde{\mathbf{m}}\mathbf{B}\tilde{\mathbf{m}}} \prod_{j=1}^{\nu_0+1} \frac{1}{(q)_{\tilde{m}_j}} \prod_{j=\nu_0+2}^{t_{n+1}} \begin{bmatrix} ((1-\mathbf{B})\tilde{\mathbf{m}})_j \\ \tilde{m}_j \end{bmatrix}_q \tag{3.11}$$

Consider next one string configurations $\mathbf{u}^{(j)}$ which ends at the position j in the zone i. We say that $\mathbf{u}^{(j)}$ is an r string if

$$\begin{aligned} & i > 0 \quad \text{and} \\ A_k &= -\frac{1}{2}\mathbf{u}_k^{(j)} \quad \text{for k in an even zone} \\ &= 0 \quad \text{otherwise} \end{aligned} \tag{3.12}$$

we say that $\mathbf{u}^{(j)}$ is an s string if

$$\begin{aligned}A_k &= -\frac{1}{2}\mathbf{u}_k^{(j)} + \frac{\nu_0 - j}{2}\delta_{k,\nu_0+1}\theta(\nu_0 - j) \;\text{ for k in an odd zone}\\ &= (k-j) \text{ for k in zone zero and } 0 \le j < \nu_0 \\ &= 0 \;\text{ otherwise}\end{aligned} \tag{3.13}$$

where $\theta(x) = 1$ if $x > 0$ and zero otherwise. With these definitions we state the following polynomial identities;

$$\begin{aligned}F(L, \mathbf{u}^{(j)}, q)_e &= B_{a,b;r;s}(L, q)\\ F(L, \mathbf{u}^{(j)}, q)_o &= B_{p'-a,b;p-r,s}(L, q)\end{aligned} \tag{3.14}$$

where for an r string

$$\begin{aligned}&s = b = s_{\min}\\ &a = l_{j+1} \text{ if} j \ne t_{i+1} \text{ and } y_{i+1} - 2y_n\delta_{i,n} \text{ if } j = t_{i+1}\\ &r = \tilde{l}_{j+1-\nu_0} \text{ if} j \ne t_{i+1} \text{ and } z_i - 2z_{n-1}\delta_{i,n} \text{ if } j = t_{i+1}\end{aligned} \tag{3.15}$$

and for an s string

$$\begin{aligned}&s = b = l_{j+1} \text{ if } j \ne t_{i+1} \text{ and } y_{i+1} - 2y_n\delta_{i,n} \text{ if } j = t_{i+1}\\ &r = r_{\min}, \quad a = s_{\min}.\end{aligned} \tag{3.16}$$

We illustrate these results for an s string in table 1 and for an r string in table 2 .

Finally we consider the two string configurations with

$$\mathbf{u} = \mathbf{u}^{(j_1)} + \mathbf{u}^{(j_2)} \;\text{ and }\; \mathbf{A} = \mathbf{A}^{(1)} + \mathbf{A}^{(2)} \tag{3.17}$$

where $\mathbf{u}^{(j_1)}$ is an r string and $\mathbf{u}^{(j_2)}$ is an s string. Then we have the following polynomial identity

$$F(L, \mathbf{u}, q)_o = B_{a,b;r,s}(L, q) \tag{3.18}$$

where

$$\begin{aligned}&s = b = l_{i_2+1} \text{ if } j_2 \ne t_{i_2+1} \;\text{ and }\; y_{i_2+1} - 2y_n\delta_{i_2,n} \text{ if } j_2 = t_{i_2+1}\\ &r = p - \tilde{l}_{j_1+1-\nu_0} \text{ if } j_1 \ne t_{i_1+1} \;\text{ and }\; p - z_{i_1} + 2z_{n-1}\delta_{i_1,n} \text{ if } j_1 = t_{i_1+1}\\ &a = p' - l_{j_1+1} \text{ if } j_1 \ne t_{i_1+1} \;\text{ and }\; p' - y_{i_1+1} + y_n\delta_{i_1,n} \text{ if } j_1 = t_{i_1+1}.\end{aligned} \tag{3.19}$$

The corresponding character identities are obtained by taking the limit $L \to \infty$ with the help of (1.4). Note in particular that for two string configurations we only obtain polynomial identities for the odd case. The possibility of character identities for two strings in the even case is considered in sec. 5.

The proofs of these results is given by generalizing the methods of [29]. The details will be published elsewhere.

4. Characters for $p' < 2p$.

When $p' < 2p$ we have $\nu_0 = 0$ and the counting problem (2.18) ceases to make sense because the inhomogeneous term is located in the zone zero and now there are no variables in zone zero. In the XXZ chain (2.1) we see from (2.2)that the regime $p < p' < 2p$ corresponds to $\Delta > 0$ and since the XXZ chain has the symmetry $H_{XXZ}(\Delta) = -H_{XXZ}(-\Delta)$ the counting problem for $\pm\Delta$ are the same. Thus we consider the relation between $M(p,p')$ and $M(p'-p,p')$ which is obtained by the transformation $q \to q^{-1}$ in the finite L polynomials $F(L,\mathbf{u},q)$ and $B_{a,b;r,s}(l,q)$. This transformation is a manifestation of the two related constructions with the fractional levels (1.6) and (1.7). However to implement the transformation it is mandatory that the $L \to \infty$ limit be taken only in the final step.

Consider first $B^{(p,p')}_{a,s,;r,s}(L,q)$ (3.1) where we have made the dependence on p and p' explicit. Then if we note that from the definition (1.2)

$$\begin{bmatrix} n+m \\ m \end{bmatrix}_{q^{-1}} = q^{-nm} \begin{bmatrix} n+m \\ m \end{bmatrix}_q \tag{4.1}$$

we find

$$B^{(p.p')}_{a,s;r,s}(L,q^{-1}) = q^{-\frac{L^2}{4}} q^{(\frac{a-s}{2})^2} B^{(p'-p,p')}_{a,s;a-r,s}(L,q). \tag{4.2}$$

Similarly we consider $q \to q^{-1}$ in $F(L,q)$ and find

$$F(L,\mathbf{u},q^{-1})_i = q^{-\frac{L^2}{4}-\Delta} \sum_{\mathrm{i-restrictions}} q^{-\frac{1}{2}\mathbf{mMm}+\mathbf{mA}'} \prod_{j=1}^{t_{n+1}} \begin{bmatrix} ((1+\mathbf{M})\mathbf{m} + \frac{\mathbf{u}}{2} + \frac{L}{2}\mathbf{e}_1)_j \\ m_j \end{bmatrix}_q \tag{4.3}$$

where the i-restrictions are given in sec. 3, Δ is the normalization constant, $\mathbf{M}$ is the matrix of coefficients of (2.18) and (2.27) (where the antisymmetric elements may be set equal to zero, and

$$\begin{aligned} \mathbf{A}'_k &= \mathbf{u}_k - \mathbf{A}_k \theta(k-\nu_o) + \frac{\nu_0 - j_1}{2}\delta_{k,\nu_0+1}\theta(\nu_0 - j_1) \\ \mathbf{u} &= \mathbf{u}^{(j_1)} + \mathbf{u}^{(j_2)} \quad \text{where} \quad j_1 < j_2 \end{aligned} \tag{4.4}$$

This mapping of $\mathbf{A} \to \mathbf{A}'$ means that the parity rules (3.12)-(3.13) change such that the r rule becomes the s rule and *vice versa.* With this definition an r (s) string remains an

r (s) string however in this case the r string may be in zone zero. Since the L dependent factor in the transform of both $B^{(p,p')}_{a,s;r,s}(L,q)$ and $F(L,\mathbf{u},q)$ are the same we derive

$$q^{\tilde{\Delta}} \sum_{\mathbf{m},\text{restrictions}} q^{-\frac{1}{2}\mathbf{m}\mathbf{M}\mathbf{m}+\mathbf{m}\mathbf{A}'} \prod_{j=1}^{t_{n+1}} \begin{bmatrix} ((1+\mathbf{M})\mathbf{m}+\frac{\mathbf{u}}{2}+\frac{L}{2}\mathbf{e}_1)_j \\ m_j \end{bmatrix}_q = B^{p'-p,p'}_{a,s,a-r,s}(L,q). \quad (4.5)$$

Making use of (1.4) we take the L $\to \infty$ limit and obtain the fermionic representations for the characters of $M(p'-p,p')$. As an example we note

$$q^{\tilde{\Delta}} \sum_{m_i=even} q^{-\frac{1}{2}\mathbf{m}\mathbf{M}\mathbf{m}} \frac{1}{(q)_{m_1}} \prod_{j=2}^{t_{n+1}} \begin{bmatrix} ((1+\mathbf{M})\mathbf{m})_j \\ m_j \end{bmatrix}_q = \chi^{p'-p,p'}_{s_{\min}-r_{\min},s_{\min}}. \quad (4.6)$$

From the physical point of view the map $M(p,p') \to M(p'-p,p')$ can be viewed as a particle-hole transformation.

We conclude this section by noting that the polynomials $B^{p'-p,p'}_{a,s;a-r,s}(L,q)$ introduced here have an explicit connection with the polynomials $D^{(k)}_L(s,a,a\pm 1)$ of Forrester and Baxter (equation 2.3.17 of [56]). Ignoring an overall constant the relation is

$$\begin{aligned} B^{p'-p,p'}_{a,s;a-r,s}(L,q) &= D^{(k)}_L(s,a,a+1) \text{ if the the r string ends in an odd zone} \\ &= D^{(k)}_L(s,a,a-1) \text{ if the r string ends in an even zone} \end{aligned} \quad (4.7)$$

and

$$\begin{aligned} k = \left[\frac{ap}{p'}\right] &= r \quad \text{if the r string ends in an odd zone} \\ &= r-1 \text{ if the r string ends in an even zone} \end{aligned} \quad (4.8)$$

where $[x]$ denotes the integer part of x. Thus we may say that the results of this paper provide the fermionic q-series form of the path counting version of the Rogers-Ramanujan identities proven in [56].

5. Discussion.

In the treatment of the characters given above we started with $M(p,p')$ for $p' > 2p$ and used the duality transformation on the finite L polynomials to compute the polynomials and the characters for $M(p'-p,p')$. There is, however, another relation between different models which was found in several special cases by Foda and Quano [31]. In this approach we start with the finite L polynomials for the model $M(p,p')$ and add one more zone to the

continued fraction decomposition by means of the Andrews-Bailey transformation [57]-[60]. This transformation produces the characters of the related minimal model with $\tilde{p}' > 2\tilde{p}$ but does not produce the finite L polynomials. We have checked that our results are consistent with the Andrews-Bailey transformation. More explicitly, if one starts , for instance, with $p, p', r = l_j, s = s_{\min}$ and takes $k-1$ steps along the so-called Bailey chain then one obtains a new model with $\tilde{p} = p',\ \tilde{p}' = p+kp',\ \tilde{s} = \tilde{l}_j + kl_j,\ \tilde{r} = s_{\min}$. It is tempting to interpret the Andrews-Bailey transformation in terms of a renormalization group flow which connects different minimal models.

We also note that the duality transformation appears to formally connect the unitary minimal model $M(p, p+1)$ with the model $M(1, p+1)$. This is indeed a correct transformation on the finite lattice. However as pointed out in [50] the limit $L \to \infty$ needed for the characters does not exist unless further factors of L are removed. This then leads to the parafermionic characters of Lepowsky and Primc [18] which are of the form of (1.1) with $\mathbf{u} = \infty$ where there are additional restrictions on the summation variables which may be interpreted as Z_N charges. This is a sort of restriction which is quite different from the even (odd) restrictions found above.

This remark leads one to ask whether or not there are other choices of the independent variables which can be made in the fermionic deformation procedure of sections 3 that will also lead to valid character formulae of some other models. One natural candidate is the model of fractional level parafermions [61] where one would speculate that all of the n_j should be chosen as independent variables. But if this is in fact correct it should be just one of many possible truncations that lead to fermionic sum formula. We also, point out that the fermionic representations of characters contain information about integrable off critical deformations of conformal field theory. Further discussion of these questions will be pursued elsewhere.

Finally we remark that there are more general relations between fermi sums and the bosonic Rocha-Caridi character formulae than the ones studied in this paper. For example, in this paper we have restricted our attention to character formulae obtained from a fermionic counting problem which is solved in terms of a single boson counting formula (2.26). A particularly interesting generalization of this are two string configurations for $m_{t_{n+1}}$ even. Then the fermionic counting function $F(L, \mathbf{u})_e$ is not solved by the single boson problem as in (2.33). Nevertheless as long as one of the two inhomogeneous terms

is in the last zone a computer study reveals that when we take the $L \to \infty$ limit we still have the single term character identity for $p' > 2p$

$$\lim_{L\to\infty} F(L, \mathbf{u}^{(j_1)} + \mathbf{u}^{(j_2)}, q)_e = \ \chi_{r,s}^{(p,p')}(q) \tag{5.1}$$

where j_1 is an r string and j_2 is an s string with

$$\begin{aligned} s &= l_{i_2+1} \text{ if} j_2 \neq t_{i_2+1} \text{ and } y_{i_2+1} - 2y_n\delta_{i_2,n} \text{ if } j_2 = t_{i_2+1} \\ r &= \tilde{l}_{j_1+1-\nu_0} \text{ if } j_1 \neq t_{i_1+1} \text{ and } z_{i_1} - 2z_{n-1}\delta_{i_1,n} \text{ if } j_1 = t_{i_1+1}. \end{aligned} \tag{5.2}$$

Let is now recall that the s and r strings are associated with $SU_{q_+}(2)$ and $SU_{q_-}(2)$ respectively. Then it is natural to interpret the breakdown of (5.1) which occurs when neither of the strings is in the last zone as a manifestation of the "Drinfeld twist" type of interaction between the two quantum groups. In general the relation of fermionic sums to bosonic sums is a matrix relation where one bosonic Rocha- Caridi character is given in terms of a linear combination of fermionic sums. Many examples of this linear combination phenomena have been seen [2]-[4], [31]. The investigation of this general theory for the models $M(p,p')$ will be the subject of future investigations.

Table1.

We illustrate the rules for the restrictions on m_j, the linear terms, and the vector of inhomogeneous terms $\mathbf{u}$ of a single s string for the case $p = 9, p' = 31$ where $\nu_0 = 2, \nu_1 = 2, \nu_2 = 2$. Here $r_{\min} = 2$ and $s_{\min} = 7$. The positions 1 and 2 are in zone 0, 3 and 4 are in zone 1 and 5 and 6 are in zone 2. We indicate by j what we mean by the position of the inhomogeneous term. The difference in this definition for j in the interior of the zone and j at the boundary of the zone at 2 and 4 are clearly seen. We have shown the seven positions of the string when $r = r_{\min}$ for the character where m_6 is even.

j	m_3	m_4	m_5	m_6	u_1	u_2	u_3	u_4	u_5	u_6	linear term	r	s
0	o	e	e	e	0	-1	0	-1	0	0	$n_1 + 2n_2 + (2m_3 + m_4)/2$	2	1
1	o	e	e	e	1	-1	0	-1	0	0	$n_2 + (m_3 + m_4)/2$	2	2
2	o	e	e	e	0	0	0	-1	0	0	$m_4/2$	2	3
3	o	e	e	e	0	0	1	-1	0	0	$(-m_3 + m_4)/2$	2	4
4	e	e	e	e	0	0	0	0	0	0	none	2	7
5	o	o	e	e	0	0	0	0	1	0	none	2	10
6	o	e	o	e	0	0	0	0	0	1	none	2	17

Table 2.

We illustrate the rules for the restrictions on m_j, the linear terms, and the vector of inhomogeneous terms $\mathbf{u}$ of a single r string for same case considered in table 1. We have shown the four positions of the string when $r = r_{\min}$ for the character where m_6 is even.

j	m_3	m_4	m_5	m_6	u_1	u_2	u_3	u_4	u_5	u_6	linear term	r	s
3	o	e	e	e	0	0	1	-1	0	0	none	1	7
4	e	e	e	e	0	0	0	0	0	0	none	2	7
5	o	o	e	e	0	0	0	0	1	0	$-m_5/2$	3	7
6	o	e	o	e	0	0	0	0	0	1	$-m_6/2$	5	7

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